The Categorification of a Symmetric Operad is Independent of Signature

Miles Gould
University of Glasgow

November 28, 2007

Abstract

Given a symmetric operad P, and a signature (or generating sequence) Φ for P, we define a notion of the *categorification* (or *weakening*) of P with respect to Φ. When P is the symmetric operad whose algebras are commutative monoids, with the standard signature, we recover the notion of symmetric monoidal categories. We then show that this categorification is independent (up to equivalence) of the choice of signature.

1 Introduction

In [2], Leinster showed how to form a categorified version of the theory of monoids, starting from any signature for that theory, and showed that for all such signatures, the categorified version was equivalent to the classical theory of monoidal categories. One might ask how far this result generalizes: in the present paper, we show that it can be extended to any theory whose models are algebras for a symmetric operad. It is believed that these theories are the "linear" ones, i.e. those which can be presented by means of equations whose variables appear exactly once on each side, though possibly not in the same order.

Section 2 covers some background material: readers familiar with the theories of factorization systems and operads can skip most of this, with the possible exception of Lemmas 2.12 and 2.19. Section 3 covers the definition of the categorification of a linear theory, and proves that this categorification is independent of the choice of signature. Section 4 uses this definition to explicitly calculate the categorification of the theory of commutative monoids with their standard signature, and shows that this is exactly the classical theory of symmetric monoidal categories. Section 5 discusses how our result relates to the 2-categorical notion of pseudo-algebras for a 2-monad, and section 6 discusses the difficulties involved in extending our approach to general finitary theories.

An earlier version of this paper was presented at the 85th Peripatetic Seminar on Sheaves and Logic in Nice in March 2007. This version differs mainly in that

the main theorem has been expanded to include symmetric (rather than non-symmetric) operads; in other words, it has been extended from strongly regular theories to linear theories. I would like to thank Michael Batanin for suggesting that I work on this generalization. The background material is also covered in more detail.

2 Background

We start by recalling some basic notions of operad theory. For more on operads, see for instance [2] Chapter 2. We borrow the notation $f_\bullet = (f_1, f_2, \ldots, f_n)$ and $g_\bullet^\bullet = (g_1^1, \ldots, g_1^{k_1}, \ldots, g_n^1, \ldots, g_n^{k_n})$ from chain complexes. We take the set of natural numbers $\mathbb{N}$ to include 0.

Definition 2.1. A **plain operad** P is

- A sequence of sets $P_0, P_1, P_2, \ldots$
- For all $n, k_1, \ldots, k_n \in \mathbb{N}$, a function $\circ : P_n \times P_{k_1} \times \cdots \times P_{k_n} \to P_{\sum k_i}$
- A **unit element** $1 \in P_1$

satisfying the following axioms:

- *Associativity:* $f \circ (g_\bullet \circ h_\bullet^\bullet) = (f \circ g_\bullet) \circ h_\bullet^\bullet$ wherever this makes sense
- *Units:* $1 \circ f = f = f \circ (1, \ldots, 1)$ for all f.

Example 2.2. Let $\mathcal{C}$ be a monoidal category and A be an object of $\mathcal{C}$. Then there is a plain operad $\mathrm{End}(A)$, called the **endomorphism operad of** A for which $\mathrm{End}(A)_n = \mathcal{C}(A^{\otimes n}, A)$. Composition in $\mathrm{End}(A)$ is given by composition and tensoring in $\mathcal{C}$.

Definition 2.3. Let P and Q be plain operads. A **morphism of plain operads** $f : P \to Q$ is a sequence of functions $f_n : P_n \to Q_n$ commuting with the composition functions in P and Q:

$$\begin{tikzcd}
P_n \times \prod P_{k_i} \arrow[r, "f_n \times f_{k_1} \times \cdots \times f_{k_n}"] \arrow[d, "\circ"'] & Q_n \times \prod Q_{k_i} \arrow[d, "\circ"] \\
P_{\sum k_i} \arrow[r, "f_{\sum k_i}"] & Q_{\sum k_i}
\end{tikzcd}$$

$$f_1(1) = 1$$

If X is some property of functions (invertibility, say), we say that an operad morphism f is **levelwise** X if every f_n is X.

Definition 2.4. A **symmetric operad** is a plain operad P together with a left action of the symmetric group S_n on each P_n, which is compatible with the operadic composition:

$$\begin{tikzcd}
P_n \times \prod P_{k_i} \arrow[r, "\sigma\times 1\times\cdots\times 1"] \arrow[d, "\circ"'] & P_n \times \prod P_{k_i} \arrow[d, "\circ"] \\
P_{\sum k_i} \arrow[r, "\sigma\circ(1,\ldots,1)"] & P_{\sum k_i}
\end{tikzcd}$$

Example 2.5. If the monoidal category $\mathcal{C}$ in Example 2.2 is symmetric, then $\mathrm{End}(A)$ acquires the structure of a symmetric operad. The symmetric groups act by composition with the symmetry map in $\mathcal{C}$.

Definition 2.6. Let P and Q be symmetric operads. A **morphism of symmetric operads** $f : P \to Q$ is a morphism of plain operads which commutes with the actions of the symmetric groups, in the sense that the diagram

$$\begin{tikzcd}
P_n \arrow[r, "\sigma\cdot -"] \arrow[d, "f_n"'] & P_n \arrow[d, "f_n"] \\
Q_n \arrow[r, "\sigma\cdot -"] & Q_n
\end{tikzcd}$$

commutes for all $n \in \mathbb{N}$ and all $\sigma \in S_n$.

Definition 2.7. Let P be an operad (plain or symmetric). An **algebra** for P is a set A and a morphism of (the appropriate kind of) operads $(\hat{\ }) : P \to \mathrm{End}(A)$. So if $p \in P_n$, then $\hat{p}$ is a function $A^n \to A$.

This amounts to a function $h_n : P_n \times A^n \to A$ for each $n \in \mathbb{N}$, satisfying some obvious axioms.

Definition 2.8. Let A, B be algebras for P. A **morphism of algebras** $A \to B$ is a function $f : A \to B$ such that

$$\begin{tikzcd}
P_n \times A^n \arrow[r, "1\times f^n"] \arrow[d, "h_n"'] & P_n \times B^n \arrow[d, "h_n"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

commutes for all $n \in \mathbb{N}$.

We form categories **Operad** and **Σ-Operad** of plain and symmetric operads and their morphisms. Given an operad P, we form a category $\mathbf{Alg}(P)$ of P-algebras and P-algebra morphisms.

The notions of (symmetric) operads and their morphisms can be interpreted in any closed symmetric monoidal category $\mathcal{V}$ in the obvious way: we call the resulting category $\mathcal{V}$-**Operad** or $\mathcal{V}$-Σ-**Operad** as appropriate. The algebras for a $\mathcal{V}$-operad are objects of $\mathcal{V}$. We are particularly interested in the case $\mathcal{V} = \mathbf{Cat}$, with the monoidal structure given by finite products.

Definition 2.9. Let $f, g : P \to Q$ be morphisms of plain **Cat**-operads. A **transformation** $\alpha : f \to g$ is a sequence $(\alpha_n : f_n \to g_n)$ of natural transformations such that

$$\begin{tikzcd} P_n \times P_\bullet \arrow[r, bend left, "f_n \times f_\bullet"] \arrow[r, bend right, "g_n \times g_\bullet"'] \arrow[r, phantom, "\Downarrow \alpha_n \times \alpha_\bullet"] \arrow[d, "\circ"'] & Q_n \times Q_\bullet \arrow[d, "\circ"] \\ P_{\sum k_i} \arrow[r, "g_{\sum k_i}"'] & Q_{\sum k_i} \end{tikzcd} = \begin{tikzcd} P_n \times P_\bullet \arrow[r, "f_n \times f_\bullet"] \arrow[d, "\circ"'] & Q_n \times Q_\bullet \arrow[d, "\circ"] \\ P_{\sum k_i} \arrow[r, bend left, "f_{\sum k_i}"] \arrow[r, bend right, "g_{\sum k_i}"'] \arrow[r, phantom, "\Downarrow \alpha_{\sum k_i}"] & Q_{\sum k_i} \end{tikzcd} \tag{1}$$

$$(\alpha_1)_1 = 1_1, \tag{2}$$

for all $n, k_1 \dots k_n \in \mathbb{N}$.

Definition 2.10. Let $f, g : P \to Q$ be morphisms of symmetric **Cat**-operads. A **transformation** $\alpha : f \to g$ is a transformation of morphisms of plain operads such that

$$\begin{tikzcd} P_n \arrow[r, bend left, "f_n"] \arrow[r, bend right, "g_n"'] \arrow[r, phantom, "\Downarrow \alpha_n"] \arrow[d, "\sigma \cdot -"'] & Q_n \arrow[d, "\sigma \cdot -"] \\ P_n \arrow[r, "g_n"'] & Q_n \end{tikzcd} = \begin{tikzcd} P_n \arrow[r, "f_n"] \arrow[d, "\sigma \cdot -"'] & Q_n \arrow[d, "\sigma \cdot -"] \\ P_n \arrow[r, bend left, "f_n"] \arrow[r, bend right, "g_n"'] \arrow[r, phantom, "\Downarrow \alpha_n"] & Q_n \end{tikzcd} \tag{3}$$

for every $n \in \mathbb{N}$ and every $\sigma \in S_n$.

We note in passing that these notions are unrelated to the operad transformations that arise from considering operads as one-object multicategories. We refer to the 2-category of plain **Cat**-operads, their morphisms and transformations as **Cat-Operad**, and to the 2-category of symmetric **Cat**-operads, their morphisms and transformations as **Cat-Σ-Operad**.

Definition 2.11. Let A, B be algebras for some symmetric **Cat**-operad P. A **weak morphism of P-algebras** $(F, \xi) : A \to B$ is a functor $F : A \to B$ and a

sequence (ξ_n) of natural transformations

$$\begin{tikzcd} P_n \times A^n \arrow[r, "1\times F^n"] \arrow[d, "h_n"'] \arrow[dr, phantom, "\swArrow\xi_n"] & P_n \times B^n \arrow[d, "h'_n"] \\ A \arrow[r, "F"] & B \end{tikzcd}$$

satisfying the equations given in [2] Section 3.2 Fig. 3-A, and in addition some extra diagrams expressing the compatibility with the symmetric group actions. We call the category of P-algebras and weak morphisms $\mathbf{Alg}_{\mathrm{wk}}(P)$.

Recall that a **fork** is a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd}$$

such that $hf = hg$, but for which h is not necessarily a coequalizer for f, g.

Lemma 2.12. *In* ***Cat*-Σ-*Operad*,** *if* $\begin{tikzcd} P \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & Q \arrow[r, "\gamma"] & R \end{tikzcd}$ *is a fork, and γ is levelwise full and faithful, then $\alpha \cong \beta$.*

Proof. We shall construct an invertible **Cat**-Σ-operad transformation $\eta : \alpha \to \beta$. We form the η_ns as follows. For all $p \in P_n$, $\gamma\alpha(p) = \gamma\beta(p)$. Since γ is levelwise full, there exists an arrow $(\eta_n)_p : \alpha(p) \to \beta(p)$ such that $\gamma_n(p) = 1_{\gamma\alpha(p)}$. Since γ is levelwise full and faithful, this arrow is an isomorphism. Each η_n is easily seen to be natural. It remains to show that the collection $(\eta_n)_{n\in\mathbb{N}}$ forms a **Cat**-Σ-operad transformation, in other words that the equations (1), (2), and (3) hold. Since γ is levelwise full and faithful, it is enough to show that the images of both sides under γ are equal, and this is trivially true by definition of η. $\square$

Definition 2.13. Let $e : a \to b, m : c \to d$ be arrows in a category $\mathcal{C}$. We say that e is **left orthogonal** to m, written $e \perp m$, if, for all arrows $f : a \to c$ and $g : b \to d$, there exists a unique map $t : b \to c$ such that the following diagram commutes:

$$\begin{tikzcd} a \arrow[r, "\forall f"] \arrow[d, "e"'] & c \arrow[d, "m"] \\ b \arrow[r, "\forall g"'] \arrow[ur, dotted, "\exists! t"] & d \end{tikzcd}$$

Definition 2.14. Let $\mathcal{C}$ be a category. A **factorization system** on $\mathcal{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ of classes of maps in $\mathcal{C}$ such that

1. for all maps f in $\mathcal{C}$, there exist $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that $f = m \circ e$.

2. $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms, and are closed under composition with isomorphisms on both sides.

3. $\mathcal{E} \perp \mathcal{M}$, i.e. $e \perp m$ for all $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

Example 2.15. Let $\mathcal{C} = \mathbf{Set}$, $\mathcal{E}$ be the epimorphisms, and $\mathcal{M}$ be the monomorphisms. Then $(\mathcal{E}, \mathcal{M})$ is a factorization system.

Example 2.16. More generally, let $\mathcal{C}$ be some variety of algebras, $\mathcal{E}$ be the regular epimorphisms (i.e., the surjections), and $\mathcal{M}$ be the monomorphisms. Then $(\mathcal{E}, \mathcal{M})$ is a factorization system.

Let **Digraph** be the category of directed graphs and graph maps.

Example 2.17. Let $\mathcal{C} = \mathbf{Digraph}$, $\mathcal{E}$ be the maps bijective on objects, and $\mathcal{M}$ be the full and faithful maps. Then $(\mathcal{E}, \mathcal{M})$ is a factorization system.

In deference to Example 2.15, we shall use arrows like $\twoheadrightarrow$ to denote members of $\mathcal{E}$ in commutative diagrams, and arrows like $\rightarrowtail$ to denote members of $\mathcal{M}$, for whatever values of $\mathcal{E}$ and $\mathcal{M}$ happen to be in force at the time.

We will use without proof the following standard properties of factorization systems:

Lemma 2.18. *Let $\mathcal{C}$ be a category, and $(\mathcal{E}, \mathcal{M})$ be a factorization system on $\mathcal{C}$.*

1. *$\mathcal{E} \cap \mathcal{M}$ is the class of isomorphisms in $\mathcal{C}$.*
2. *The factorization in (1) is unique up to unique isomorphism.*
3. *The factorization in (1) is functorial; in other words, if the square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "f'"] & D \end{tikzcd}$$

commutes, and $f = me, f' = m'e'$, then there is a unique morphism i making

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "g"'] & \cdot \arrow[r, "m"] \arrow[d, dotted, "i"'] & B \arrow[d, "h"] \\ C \arrow[r, "e'"] & \cdot \arrow[r, "m'"] & D \end{tikzcd}$$

commute. Thus, $\mathcal{E}$ and $\mathcal{M}$ can be regarded as functors $Ar(\mathcal{C}) \to Ar(\mathcal{C})$.

4. *$\mathcal{E}$ and $\mathcal{M}$ are closed under composition.*
5. *$\mathcal{E}^{\perp} = \mathcal{M}$ and ${}^{\perp}M = \mathcal{E}$, where $\mathcal{E}^{\perp} = \{f \text{ in } \mathcal{C} : e \perp f \text{ for all } e \in \mathcal{E}\}$ and ${}^{\perp}\mathcal{M} = \{f \text{ in } \mathcal{C} : f \perp m \text{ for all } m \in \mathcal{E}\}$.*

We will also use the following fact:

Lemma 2.19. *Let $\mathcal{C}$ be a category with a factorization system $(\mathcal{E}, \mathcal{M})$. Let T be a monad on $\mathcal{C}$ and let $\overline{\mathcal{E}} = \{f \text{ in } \mathcal{C} : Uf \in \mathcal{E}\}$, $\overline{\mathcal{M}} = \{f \text{ in } \mathcal{C} : Uf \in \mathcal{M}\}$ where U is the forgetful functor $\mathcal{C}^T \to \mathcal{C}$. Suppose that T preserves $\mathcal{E}$-arrows. Then $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ is a factorization system on $\mathcal{C}^T$.*

Proof. Standard. See [1], Proposition 20.24. □

Example 2.20. Let $(\mathcal{E}, \mathcal{M})$ be the factorization system on **Digraph** described in Example 2.17 above, and let T be the free category monad. Since **Cat** is monadic over **Digraph**, this gives a factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ on **Cat** where $\overline{\mathcal{E}}$ is the bijective-on-objects functors, and $\overline{\mathcal{M}}$ is the full and faithful ones.

Example 2.21. Let $\mathcal{C} = \mathbf{Cat}^{\mathbb{N}}$, $\mathcal{E}$ be the bijective-on-objects maps, and $\mathcal{M}$ be those that are levelwise full and faithful. Since **Cat-Σ-Operad** is monadic over $\mathbf{Cat}^{\mathbb{N}}$, this gives a factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ on **Cat-Σ-Operad** where $\overline{\mathcal{E}}$ is the bijective-on-objects maps, and $\overline{\mathcal{M}}$ is the levelwise full and faithful ones.

We shall need one final piece of background:

Theorem 2.22. *If X is a set and T is a monad on $\mathbf{Set}^X$ then the regular epis in $(\mathbf{Set}^X)^T$ are the coordinate-wise surjections. In other words, the forgetful functor $U : (\mathbf{Set}^X)^T \to \mathbf{Set}^X$ preserves and reflects regular epis.*

Proof. See [1] section 20, in particular Definition 20.21 and Proposition 20.30. □

3 Categorification

Throughout, let P be a symmetric **Set**-operad. Symmetric operads are algebras for a straightforward multi-sorted algebraic theory, so (by standard arguments from universal algebra) there is a monadic adjunction

$$\begin{tikzcd} \mathbf{Set}^{\mathbb{N}} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \Sigma\text{-}\mathbf{Operad} \arrow[l, shift left, "U"] \end{tikzcd}$$

Definition 3.1. A **signature** for P is a pair (Φ, ϕ), where $\Phi \in \mathbf{Set}^{\mathbb{N}}$ and $\phi : F\Phi \to P$ is a regular epi. Where ϕ is obvious, we shall abuse notation and refer to Φ as a signature.

Example 3.2. (UP, ϵ) is a signature for P, where ϵ is the counit of the adjunction $F \dashv U$, since

$$\begin{tikzcd} FUFUP \arrow[r, shift left, "\epsilon FU"] \arrow[r, shift right, "FU\epsilon"'] & FUP \arrow[r, "\epsilon"] & P \end{tikzcd}$$

is a coequalizer diagram.

Example 3.3. The sequence $(\{e\}, \emptyset, \{.\}, \emptyset, \emptyset, \ldots)$ is a signature for the terminal symmetric operad, whose algebras are commutative monoids.

We will define a categorification of any symmetric operad P, dependent on a signature (Φ, ϕ). This will be a "weak" categorification, in the sense that derived operations which are equal in P will only be isomorphic in the categorified theory. We will then show that this is independent of our choice of signature, in the sense that the symmetric **Cat**-operads which arise are equivalent (and thus have equivalent categories of algebras).

Definition 3.4. Let (Φ, ϕ) be a signature for P.

Embed **Σ-Operad** into **Cat-Σ-Operad** via the (full and faithful) discrete category functor. Using the factorization system of Example 2.21, factor ϕ as follows in **Cat-Σ-Operad**:

$$\begin{tikzcd} F\Phi \arrow[rr, "\phi"] \arrow[dr, "b"'] & & P \\ & Q \arrow[ur, tail, "f"'] & \end{tikzcd}$$

where f is full and faithful levelwise, and b is bijective on objects. Then the **categorification of P with respect to** (Φ, ϕ) is Q. The uniqueness of Q follows from property (2) in Lemma 2.18.

Example 3.5. (This example will be explored in greater detail in Section 4). Let P be the terminal symmetric operad, whose algebras are commutative monoids. Let Φ be the standard signature for commutative monoids, i.e. a binary operation and a constant. Then the categorification Q of P with respect to Φ is the symmetric **Cat**-operad whose algebras in **Cat** are symmetric monoidal categories.

Objects of Q are permuted trees of binary and nullary nodes, and there is a unique arrow between two trees τ_1 and τ_2 if and only if they evaluate to the same operation in the theory of commutative monoids. By uniqueness, all diagrams in Q commute.

A Q-algebra, therefore, is a category C equipped with a binary functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$; an object $I \in \mathcal{C}$; and a natural transformation from a permuted composite of these functors to another if and only if they have the same number of arguments. Here a "permuted composite" is something like the functor $(A, B, C) \mapsto (A \otimes ((I \otimes C) \otimes B))$, where variables may be permuted but not repeated or dropped. Diagrams of these natural transformations will commute if all the morphisms involved have each variable appearing once on each side; but this is precisely the coherence theorem for symmetric monoidal categories given in [3] Chapter XI, Theorem 1.1. In particular, all these natural transformations will be invertible.

Definition 3.6. The **unbiased categorification** of P is the categorification arising from the signature (UP, ϵ) described in Example 3.2. Call this symmetric **Cat**-operad $\mathrm{Wk}(P)$.

Definition 3.7. An **unbiased weak P-category** is an algebra for $\mathrm{Wk}(P)$. A **weak P-functor** is a weak morphism of $\mathrm{Wk}(P)$-algebras between two unbiased weak P-categories.

Recall that **Cat-Σ-Operad** is a 2-category, so we may talk of two symmetric **Cat**-operads being equivalent.

Theorem 3.8. *Every categorification of P is equivalent as a symmetric **Cat**-operad to* $\mathrm{Wk}(P)$*.*

Proof. Let Q be the categorification of P with respect to a signature (Φ, ϕ). By the triangle identities, we have a commutative square

$$\begin{tikzcd}
F\Phi \arrow[r, "\phi"] \arrow[d, "F\overline{\phi}"'] & P \arrow[d, "1"] \\
FUP \arrow[r, "\epsilon"] & P
\end{tikzcd}$$

By functoriality of the factorization system, this gives rise to a unique map $\chi : Q \to \mathrm{Wk}(P)$ such that

$$\begin{tikzcd}
F\Phi \arrow[r, two heads] \arrow[d, "F\overline{\phi}"'] \arrow[rr, bend left, "\phi"] & Q \arrow[r, tail] \arrow[d, dotted, "\chi"] & P \arrow[d, "1"] \\
FUP \arrow[r, two heads] \arrow[rr, bend right, "\epsilon"'] & \mathrm{Wk}(P) \arrow[r, tail] & P
\end{tikzcd}$$

commutes. We wish to find a pseudo-inverse to χ.

Since **Σ-Operad** is monadic over $\mathbf{Set}^{\mathbb{N}}$, a regular epi in **Σ-Operad** is a pointwise surjection (intuitively, the fact that Φ is a signature for P means that Φ generates P, so $\phi_n : (F\Phi)_n \to P_n$ is surjective). So we may choose a section ψ_n of $\phi_n : (F\Phi)_n \to P_n$ for all $n \in \mathbb{N}$. So we have a morphism $\psi : UP \to UF\Phi$ in $\mathbf{Set}^{\mathbb{N}}$ that is a section of $U\phi$. We wish to show that

$$\begin{tikzcd}
FUP \arrow[r, "\epsilon_P"] \arrow[d, "\overline{\psi}"'] & P \arrow[d, "1"] \\
F\Phi \arrow[r, "\phi"] & P
\end{tikzcd}$$

commutes. Indeed,

$$\frac{FUP \xrightarrow{\bar{\psi}} F\Phi \xrightarrow{\phi} P}{UP \xrightarrow{\psi} UF\Phi \xrightarrow{U\phi} UP} = \frac{UP \xrightarrow{1} UP}{FUP \xrightarrow{\epsilon} P},$$

as required. This induces a map

$$\begin{tikzcd}
FUP \arrow[r, two heads] \arrow[d, "\overline{\psi}"'] \arrow[rr, bend left, "\epsilon"] & \mathrm{Wk}(P) \arrow[r, tail] \arrow[d, dotted, "\omega"] & P \arrow[d, "1"] \\
F\Phi \arrow[r, two heads] \arrow[rr, bend right, "\phi"'] & Q \arrow[r, tail] & P
\end{tikzcd}$$

We will show that ω is pseudo-inverse to χ. Now,

$$\begin{tikzcd}
\mathrm{Wk}(P) \arrow[r, tail] \arrow[d, "\omega"'] & P \arrow[d, "1"] \\
Q \arrow[r, tail] \arrow[d, "\chi"'] & P \arrow[d, "1"] \\
\mathrm{Wk}(P) \arrow[r, tail] & P
\end{tikzcd}$$

commutes. So

$$\begin{tikzcd}
\mathrm{Wk}(P) \arrow[r, shift left, "1_Q"] \arrow[r, shift right, "\chi\omega"'] & \mathrm{Wk}(P) \arrow[r] & P
\end{tikzcd}$$

is a fork. By Lemma 2.12, $\chi\omega \cong 1_{\mathrm{Wk}(P)}$, and similarly $\omega\chi \cong 1_Q$. So $Q \simeq \mathrm{Wk}(P)$ as a symmetric **Cat**-operad, as required. □

Corollary 3.9. *Let Q be a categorification of P with respect to some signature (Φ, ϕ). Then $\mathbf{Alg}_{\mathrm{wk}}(Q) \simeq \mathbf{Alg}_{\mathrm{wk}}(\mathrm{Wk}(P))$.*

Proof. By a straightforward extension of the proof in [2] Theorem 3.2.3, $\mathbf{Alg}_{\mathrm{wk}}$ is a 2-functor $\mathbf{Cat}\text{-}\mathbf{\Sigma}\text{-}\mathbf{Operad} \to \mathbf{CAT}^{\mathrm{co\ op}}$, and thus preserves equivalences. □

4 Symmetric monoidal categories

Let Q be the categorification of the terminal symmetric operad P (whose algebras are commutative monoids) with respect to the standard signature Φ given in Example 3.3 above - recall that Φ has a nullary element e and a binary element . , and is empty in all other arities. Here we prove our assertion in Example 3.5 that the algebras for Q are classical symmetric monoidal categories. More precisely, we show that for a given category $\mathcal{C}$, the Q-algebra structures on $\mathcal{C}$ are in one-to-one correspondence with the symmetric monoidal category structures on $\mathcal{C}$.

Recall that a symmetric monoidal category is a structure $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$ where

- $\mathcal{C}$ is a category,
- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- $I \in \mathcal{C}$,
- $\alpha : - \otimes (- \otimes -) \to (- \otimes -) \otimes -$,
- $\lambda : I \otimes - \to 1_{\mathcal{C}}$,
- $\rho : - \otimes I \to 1_{\mathcal{C}}$,
- $\tau : (12) \cdot \otimes \to \otimes$, where (12) is the non-identity permutation of $\{1, 2\}$,

satisfying the axioms given in [3] Chapter XI.

Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category. We will define a Q-algebra structure on $\mathcal{C}$, which we shall call $S(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau) = (\mathcal{C}, (\hat{\ }))$. Since there's a bijective-on-objects map of symmetric **Cat**-operads from $F\Phi$ to Q, we may define the action of the objects of Q on $\mathcal{C}$ by giving a **Cat**-operad map $(\hat{\ }') : F\Phi \to \mathrm{End}(\mathcal{C})$. Equivalently, we may give a map $\Phi \to U\,\mathrm{End}(\mathcal{C})$ in $\mathbf{Set}^{\mathbb{N}}$, which amounts to a choice of a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an element of $\mathcal{C}$. Let these be $\otimes$ and I. To define the actions of the morphisms of Q on $\mathcal{C}$, we need a natural transformation $\hat{\delta} : \hat{q}_1 \to \hat{q}_2$ for each arrow $\delta : q_1 \to q_2 \in Q_n$. By construction of Q, there is such an arrow whenever q_1 and q_2, considered as derived operations of the theory of commutative monoids, evaluate to the same operation. By standard properties of commutative monoids, this means that we want a natural transformation $\hat{q}_1 \to \hat{q}_2$ iff q_1 and q_2 take the same number of arguments. The coherence theorem for classical symmetric monoidal categories ([3] XI.1) gives us exactly this (via the "canonical" maps), and ensures that the composite and tensor of two such canonical maps are canonical, i.e. that we have a well-defined map of **Cat**-operads $Q \to \mathrm{End}(\mathcal{C})$. Hence, $(\mathcal{C}, (\hat{\ }))$ is a well-defined Q-algebra.

Now, let $\mathcal{C}$ be a Q-algebra, with map $(\hat{\ }) : Q \to \mathrm{End}(\mathcal{C})$. We shall construct a symmetric monoidal category $R(\mathcal{C}, (\hat{\ })) = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$. Take

- $\otimes = (\hat{.})$
- $I = \hat{e}$
- $\alpha = \hat{\delta}_1$, where $\delta_1 : -.(-.-) \to (-.-).-$ in Q,
- $\lambda = \hat{\delta}_2$, where $\delta_2 : e.- \to -$,
- $\rho = \hat{\delta}_3$, where $\delta_3 : -.e \to e$,
- $\tau = \hat{\delta}_4$, where $\delta_4 : (12) \cdot (-.-) \to (-.-)$.

Each δ_i is uniquely defined by its source and target, since each Q_n is a poset. Because there's at most one map $q_1 \to q_2$ for any $q_1, q_2 \in Q_n$, all diagrams involving these commute. In particular, the axioms for a symmetric monoidal category are satisfied. So $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$ is a symmetric monoidal category.

Now, let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$ be a symmetric monoidal category. We wish to show that $RS(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau) = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau)$. Let $RS(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \tau) = (\mathcal{C}, \otimes', I', \alpha', \lambda', \rho', \tau')$. Their underlying categories are equal, both being $\mathcal{C}$. Furthermore,

$$\begin{array}{ccccl} \otimes' & = & \hat{.} & = & \otimes \\ I' & = & \hat{e} & = & I \\ \alpha' & = & \hat{\delta}_1 & = & \alpha\text{, the unique canonical map of the correct type} \\ \lambda' & = & \hat{\delta}_2 & = & \lambda \\ \rho' & = & \hat{\delta}_3 & = & \rho \\ \tau' & = & \hat{\delta}_4 & = & \tau \end{array}$$

Conversely, let $(\mathcal{C}, (\hat{\ }))$ be a Q-algebra, and let $SR(\mathcal{C}, (\hat{\ })) = (\mathcal{C}', (\hat{\ }'))$. Does $(\mathcal{C}, (\hat{\ })) = (\mathcal{C}', (\hat{\ }'))$? Their underlying categories are the same. As above, $(\hat{\ }')$ is determined on objects by the values it takes on $.$ and e: these are $\otimes = \hat{.}$ and $I = \hat{e}$ respectively. So $(\hat{\ }') = (\hat{\ })$ on objects. If $\delta : \tau_1 \to \tau_2$, then $\hat{\delta}'$ is the unique canonical map from $\hat{\tau}_1' \to \hat{\tau}_2'$, which, by an easy induction, must be $\hat{\delta}$. So $(\hat{\ }') = (\hat{\ })$, and hence $(\mathcal{C}, (\hat{\ })) = SR(\mathcal{C}, (\hat{\ }))$.

5 Relation to pseudo-algebras

Another approach to categorification of theories might be to promote the induced monad T on **Set** to a 2-monad on **Cat** via the discrete category functor, and then to take pseudo-algebras for this 2-monad. It is well-known to the 2-categorical cognoscenti that this is equivalent to our construction for strongly-regular theories. However, for linear theories, this is not the case: for instance, the pseudo-algebras for the free commutative monoid monad are the strictly symmetric monoidal categories, i.e. those symmetric monoidal categories for which the symmetry map τ_{AB} is an identity for all A, B.

6 General algebraic theories

In light of Lemma 2.19, one might ask the following question. It is easy to extend the definition of operad (as was done, for instance, in [4]) so that every finitary algebraic theory is represented by one of these more general operads. The definition of categorification presented in this paper extends straightforwardly to this situation, to give a categorification of *any* algebraic theory. Is this definition sensible?

Unfortunately, it isn't. The two classes of arrows in our factorization system are, again, the levelwise bijective-on-objects arrows and the levelwise full-and-faithful arrows; this corresponds to a categorified theory in which *all* diagrams commute, but this is not what we want. For instance, in the theory of commutative monoids, this would imply that the diagram

$$A \otimes A \underset{\tau_{A,A}}{\overset{1}{\rightrightarrows}} A \otimes A$$

commutes, where τ is the symmetry map. This is not the case for most interesting symmetric monoidal categories. Moreover, most symmetric monoidal categories are not even equivalent to one with this property.

References

[1] Jiří Adámek, Horst Herrlich, and George E.Strecker. *Abstract and Concrete Categories: The Joy of Cats*. Second edition, 2004.

[2] Tom Leinster. *Higher Operads, Higher Categories*. Cambridge University Press, 2003.

[3] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, second edition, 1998.

[4] S.N. Tronin. Abstract clones and operads. *Siberian Mathematical Journal*, 43(4):746–755, Jul 2002.